Hertz potentials, peeling, and the Cauchy problem

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Introduction

- Context : Study of the nonlinear stability of Kerr black holes.
- Previous result : Minkowski nonlinear stability ('91), relies heavily on decay result for higher spin fields ('89) on flat background.
- Question : develop alternate methods to study the asymptotics of higher spin fields
- Exploit symmetries of the spacetime/higher spin fields (Maxwell fields and linearized gravity).
- Structure of potentials (Hertz, Debye, etc) strongly tight to the structure of the space-time.

Introduction : Hertz potentials

- Penrose (63') : representations of massless spin-s fields on flat space-time : local representation by a potential of order 2s, satisfying a wave equation.
- Penrose proved peeling from a decay assumption on χ.
- (Cohen-Kegeles 76) : on Kerr black holes :

$$F = \overline{\mathrm{d}\delta} G, \text{ G solution of a wave equation}$$

  is an uncharged solution of the Maxwell equations.
- Conjecture : Any Maxwell fields can be written as :

$$F = F_{Coulomb} + \overline{\mathrm{d}\delta} G$$

- Conjecture : in this situation, $\overline{\mathrm{d}\delta} G$ radiates/decays, under suitable assumptions.

Today's talk's framework and purpose

- Background : flat space-time.
- Cauchy problem for massless spin-s fields of arbitrary spin but especially Maxwell (spin 1) and linearized gravity (spin 2).
- Construct a potential satisfying a wave equation, whose initial data lie in a Sobolev space insuring good decay properties.
- Deduce decay/peeling properties.
- Important result : Christodoulou-Klainerman '89 on linear fields.

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Scalar fields on flat background

- Restrict our attention to fields on Minkowski space :

$$(\mathbb{R}^4, \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2)$$

- Consider the Cauchy problem for the wave equation :

$$\begin{cases} \Box\phi = 0 \\ \phi|_{t=0} = f \in H^k_\sigma(\mathbb{R}^3) \\ \partial_t\phi|_{t=0} = g \in H^{k-1}_{\sigma-1}(\mathbb{R}^3) \end{cases}$$

- Purpose : How does this field decay ?

Decay of solutions of the linear wave equation

- Background : flat space-time.
- Obtained by energy estimates (Klainerman 83-87, $\sigma = -\frac{3}{2}$) or by conformal compactification (Penrose 65, stronger assumptions on the initial data, $\sigma = -2$).
- There exist for arbitrary weights (Asakura '86, d'Ancona-Georgiev-Kubo '01, Szpak '08).
- Obtain decay estimates in two directions :
  - Interior decay : along time directions ($t > 3r$) ;
  - Exterior decay : along null directions ($\frac{t}{3} < r < 3t$).

Decay of solutions of the linear wave equation

Theorem (Klainerman)

Let $s_0 \geq 2$. Let u be a solution of the wave equation with initial date in $H^{s_0}_{-\frac{3}{2}}(\mathbb{R}^3) \times H^{s_0-1}_{-\frac{5}{2}}(\mathbb{R}^3)$. Then

1. *for $t > 3r$*

$$|\phi(t,x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2},s_0}}{< t >^{\frac{3}{2}}},$$

2. *for $\frac{r}{3} < t < 3r$:*

$$|\phi(t,x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2},s_0}}{< u >^{\frac{1}{2}} < v >^{1}},$$

Decay of solutions of the linear wave equation 2

Theorem (Klainerman)

Let $s_0 \geq 2$, $(j,k,l) \in \mathbb{N}^3$. Let u be a solution of the wave equation with data in $H^{s_0+j+k+l}_{-\frac{3}{2}}$. Then

1. *for $t > 3r$*

$$|\nabla^j \phi(t,x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2},s_0+j}}{< t >^{\frac{3}{2}+j}},$$

2. *for $\frac{r}{3} < t < 3r$:*

$$|\partial_u^j \partial_v^k \nabla_{\mathbb{S}^2}^l \phi(t,x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2},s_0+j+k+l}}{< u >^{\frac{1}{2}+j} < v >^{1+k+l}},$$

$u = t - r$ and $v = t + r$.

Decay result for arbitrary weights

- For arbitrary σ ?
- $\phi|_{t=0} = f \in H^k_\sigma$ and $\partial_t \phi|_{t=0} = g \in H^{k-1}_{\sigma-1}$, $k \geq 3$:

$$f(x) \lesssim < r >^{\sigma} \|f\|_{2,\sigma} \quad \text{and} \quad g(x) \lesssim < r >^{\sigma-1} \|g\|_{2,\sigma-1}$$

- Integral representation :

$$\phi(t,x) = \frac{1}{4\pi}\left(\int_{\mathbb{S}^2} t\left(g(x+t\omega) + \partial_\omega f(x+t\omega)\right) + f(x+t\omega)\mathrm{d}\mu_{\mathbb{S}^2}\right)$$

- Asymptotic behavior is given by :

$$J_\sigma = \int_{\mathbb{S}^2} < |x+t\omega| >^{\sigma} \mathrm{d}\mu_{\mathbb{S}^2}.$$

Decay result for arbitrary weights

- $J_\sigma =$
$$\begin{cases} 8\pi \dfrac{<u>^{2+\sigma} - <v>^{2+\sigma}}{(2+\sigma)(<u>^2 - <v>^2)} & \text{if } \sigma \neq -2 \text{ and } <u> \neq <v>, \\ 8\pi \dfrac{\log\left(\frac{<u>}{<v>}\right)}{(<u>^2 - <v>^2)} & \text{if } \sigma = -2 \text{ and } <u> \neq <v>, \\ 4\pi <v>^\sigma & \text{if } <u> = <v>. \end{cases}$$
- For the full solution, combine J_σ and $J_{\sigma-1}$, hence the discussion arises on $\sigma = -1$.
- For higher order derivatives, one use commutations with $\Box$.

Decay for arbitrary weights

Proposition

If (f,g) in $H^m_\sigma \times H^{m-1}_{\sigma-1}$, $m \geq j+k+l+3$, one denotes :

$$I_\sigma = ||(f,g)||_{H^{j+k+l+3}_\sigma \times H^{j+k+l+2}_{\sigma-1}}$$

then :

$$|\partial_u^k \partial_v^l \nabla^m_{S^2} \phi| \leq C I_\sigma \begin{cases} < u >^{1+\sigma-k} < v >^{-1-l-m} & \text{if } \sigma < k-1 \\ \dfrac{\log < v > - \log < u >}{< v >^{l+m} (< v > - < u >)} & \text{if } \sigma = k-1 \\ < v >^{\sigma-l-m-k} & \text{if } \sigma > k-1 \end{cases} .$$

Problem

- Give a proper analytic framework to Penrose's representation of massless fields of spin 2s :

$$\phi_{A\ldots F} = \underbrace{\nabla_{AA'} \ldots \nabla_{FF'}}_{2s-\text{derivatives}} \xi^{A'\ldots F'}, \text{where } \Box\xi = 0.$$

- Proper analytic representation : Cauchy problem for the field to a Cauchy problem for the potential + control of the norm of the initial data of the potential.
- Cases of interest : Maxwell and linearized gravity on flat background.
- Methods : elementary elliptic theory.

Maxwell equations

- Geometric background : Minkowski background $(\mathbb{R}^4, \eta)$.
- Consider the Faraday (skew-symmetric) tensor (2-form) : F.
- Link with electric and magnetic fields (1-forms on $\mathbb{R}^3$) : $T^a = (1,0,0,0)$:

$$E = F(T, \bullet) \text{ and } B = (\star F)(T, \bullet)$$

  where $\star F$ is the Hodge dual.
- Maxwell equations :

$$\bar{\mathrm{d}} F = 0 \text{ and } \bar{\delta} F = 0$$
$$\nabla_{[a} F_{bc]} = 0 \text{ and } \nabla^a F_{ab} = 0$$

Cauchy problem for the Maxwell equations

- Hyperbolic system of order 1 with 6 real unknowns with geometric constraints on the initial data.
- $\begin{cases} (\overline{\mathrm{d}} + \overline{\delta})F = 0 \\ F|_{t=0} \in H^k_\sigma(\mathbb{R}^3, \Lambda^2) \end{cases}$
- Geometric constraints on the initial data :

$$D^a E_a = D^a B_a = 0$$

- Purpose : construct initial data for G such that F of the form :

$$F = \overline{\mathrm{d}\delta} G$$

Construction of a potential for the Maxwell field

- Assume $F = \overline{\mathrm{d}\delta} G$.
- Restrict to $t = 0$:

$$\begin{aligned} E &= -\delta \mathrm{d} H - \delta \star \partial_t K \\ B &= -\delta \mathrm{d} K + \delta \star \partial_t H \end{aligned}$$

- Solve for a given set of initial data with :

$$H = K = 0.$$

- Take $E, B \in H^k_\sigma$ in the image of Δ :

$$E = (\mathrm{d}\delta + \delta\mathrm{d})\,\tilde{H}$$
$$B = (\mathrm{d}\delta + \delta\mathrm{d})\,\tilde{K}$$

with $\tilde{H}, \tilde{K} \in H^{k+2}_{\sigma+2}$.

Construction of a potential

- Use the geometric constraints :

$$\delta E = 0 \quad \Rightarrow \quad E = \delta \left(\mathrm{d}\tilde{H}(+\mathrm{sth}) \right)$$
$$\delta B = 0 \quad \Rightarrow \quad B = \delta \left(\mathrm{d}\tilde{K}(+\mathrm{sth}) \right)$$

- Take as initial data for the potential : $(0, - \star \mathrm{d}\tilde{H})$ and $(0, \star \mathrm{d}\tilde{K})$.
- Conditions to admit a potential :

$$E, B \in H^k_\sigma(\Lambda^1) \perp \mathrm{Ker}(\Delta) \cap L^2_{-3-\sigma}(\Lambda^1)$$

Construction of a potential

Lemma

Let $\sigma < -2$. If E, B in $H^k_\sigma(\Lambda^1)$ satisfy the constraints equation,

$$\delta E = \delta B = 0,$$

there exist E_1, B_1 such that

$$E + \delta \star E_1, B + \delta \star B_1 \perp_{L^2} \ker(\Delta) \subset L^2_{-3-\sigma}.$$

Proof : Projection + rescaling

Existence of potentials for Maxwell fields

Proposition

Let σ in $\mathbb{R} \setminus \mathbb{Z}_-$ and $s_0 \geq 3$.
Let E_0, B_0 be two solutions of the constraints in $H^{s_0}_{\sigma}$.
Then there exist two 2-forms (G_0, G_1) in $H^{s_0+2}_{\sigma+2} \times H^{s_0+1}_{\sigma+1}$ such that :

$$||G_0||^2_{s_0+2,\sigma+2} + ||G_1||^2_{s_0+1,\sigma+1} \leq C||F_0||^2_{s_0,\sigma}$$

and :

$$F = \overline{d\delta} G$$

where G is the solution of the wave equation $(\overline{d\delta} + \overline{\delta d})G = 0$ with initial data (G_0, G_1).

Linearized gravity – tensor version

- W_{abcd}, a 4-tensor satisfying the symmetries of the Weyl spinor.
- Consider the Cauchy problem :

$$\left\{ \begin{array}{l} \nabla^a W_{abcd} = 0 \\ W_{abcd} = \psi_{abcd} \in H^k_\sigma + \text{ constraints} \end{array} \right. .$$

- Introduce E, B :

$$\begin{array}{ccc} E_{cd} = T^a T^b W_{abcd} & \text{and} & B_{cd} = T^a T^b (\star W_{abcd}) \\ D^a E_{ab} = 0 & & D^a B_{ab} = 0 \end{array} .$$

- Hyperbolic system of order 1 of 10 unknowns.
- Unfortunately, no simple tensor notations :

$$W = \nabla \underbrace{\nabla \qquad \overbrace{\nabla\nabla\xi}^{\text{Bergman potential}} \qquad}_{\text{Lanczos potential}} .$$

Linearized gravity – Spinor version

- ϕ_{ABCD}, a totally symmetric spinor.
- Consider the Cauchy problem :

$$\left\{ \begin{array}{l} \nabla^{AA'}\phi_{ABCD} = 0 \\ \phi_{ABCD} = \psi_{ABCD} \in H^k_\sigma + \text{ constraints } D^{AB}\psi_{ABCD} = 0 \end{array} \right. .$$

- Hyperbolic system of order 1 of 5 complex unknowns.
- In spinors, the potential writes :

$$\phi_{ABCD} = \nabla_{AA'} \underbrace{\nabla_{BB'} \overbrace{\nabla_{CC'}\nabla_{DD'}\xi^{A'B'C'D'}}^{\text{Bergman potential}}}_{\text{Lanczos potential}} .$$

Sketch of the proof

- Principle of the proof is the same except that one has to work with Δ^2.
- Only important change : integrability condition on E_{ab} and B_{ab} :

Proposition

If E_{ab} satisfies the constraint,

$$\nabla^a E_{ab} = 0 \text{ or } \delta_2 E = 0$$

then there exists a G_{ab} such that

$$\mathcal{R}(G) = E$$

where $\mathcal{R}$ is the linearized Cotton-York tensor.

Integration of 2-tensors

- Conformal rigidity : the deformation of a metric g_0, $\{g_t\}_t$, is conformally rigid iff there exist a family of diffeomorphisms ϕ_t and functions u_t such that :

$$\phi_t^\star g_0 = e^{u_t} g_t$$

  with $\phi_0 = Id$ and $u_0 = 0$.
- The conformal Killing equation :

$$L_X g_0 - \frac{1}{3}\mathrm{Tr}(L_X g_0) g_0 = h \text{ or } 2D_{(AB}X_{CD)} = h_{ABCD}$$

  can only be integrated provided that :

$$0 = \epsilon_{ab}{}^{d}\mathcal{R}(h)_{dc} = 2D_{[a}\sigma_{b]c} \text{ where}$$
$$\sigma_{ab} = D_{(a}D^c h_{b)c} - \tfrac{1}{2}\Delta h_{ab} - \tfrac{1}{4} g_{ab} D^c D^d h_{cd}.$$

  $\mathcal{R}$ is the linearized Cotton-York tensor.

De Rham and Gasqui-Goldschmidt complexes

- Previous works : Gasqui-Golschmidt '84 ; Beig '97
- Solving

$$\delta \star f = \omega \left(\omega \in \Lambda^1\right) \text{ or } L_X g_0 - \frac{1}{3}\mathrm{Tr}(L_X g_0) g_0 = h$$

requires that the lhs satisfy constraints.

- Constraints are solved by the differential complexes :

$$\begin{array}{ccccccc} C^\infty(M,\mathbb{R}) & \xrightarrow{d} & \Lambda^1 & \xrightarrow{\delta\star} & \Lambda^1 & \xrightarrow{\delta} & C^\infty(M,\mathbb{R}) \\ \Lambda^1(M) & \xrightarrow{L} & S^2_0(M,g) & \xrightarrow{\mathcal{R}} & S^2_0(M,g) & \xrightarrow{\delta_2} & \Lambda^1(M) \end{array}$$

L : conformal Killing operator, δ_2 : divergence on 2 tensors.

Existence of a potential for spin-2 fields

Proposition

Let σ in $\mathbb{R} \setminus Z_-$ and $s_0 \geq 3$.
Let ψ_{ABCD} be a solution to the constraints in $H^{s_0}_\sigma$.
Then there exists (ξ_0, ξ_1) in $H^{s_0+4}_{\sigma+4} \times H^{s_0+3}_{\sigma+3}$ such that :

$$||\xi_0||^2_{s_0+2,\sigma+4} + ||\xi_1||^2_{s_0+1,\sigma+3} \leq C||\psi||^2_{s_0,\sigma}$$

and

$$\phi_{ABCD} = \nabla_{AA'} \underbrace{\nabla_{BB'} \overbrace{\nabla_{CC'} \nabla_{DD'} \xi^{A'B'C'D'}}^{\textit{Bergman potential}}}_{\textit{Lanczos potential}}.$$

where ξ is the solution of the wave equation $\Box \xi = 0$ with initial data (ξ_0, ξ_1).

- Purpose : Study the asymptotic behavior of spin s fields satisfying the Dirac equation on flat background by methods which could be extended to Kerr background.
- Work of reference : Christodoulou-Klainerman ('89)
- Here : derive the same kind of decay result using representation of fields using potentials by reducing the tensor equation to a scalar wave equation.

Asymptotics for the Maxwell equations

Weight – ID	$-\frac{7}{2}$	$-\frac{5}{2}$ (ABJ)	$-\frac{5}{2}$ (CK)
Weight – ID potential	$-\frac{3}{2}$	$-\frac{1}{2}$ (ABJ)	X
Interior decay $t > 3r$			
	$t^{-\frac{7}{2}}$	$t^{-\frac{5}{2}}$	$t^{-\frac{5}{2}}$
Exterior decay $\frac{t}{3} < r < 3t$			
$F(\partial_u, e_{\mathbb{S}^2})$, $\underline{\alpha}$, ϕ_{-1}, ϕ_2	$u^{-\frac{5}{2}} v^{-1}$	$u^{-\frac{3}{2}} v^{-1}$	$u^{-\frac{3}{2}} v^{-1}$
$F(\partial_v, \partial_u)$, ρ, $F(e_{\mathbb{S}^2}, e_{\mathbb{S}^2})$, σ, ϕ_0, ϕ_1	$u^{-\frac{1}{2}} v^{-2}$	$u^{-\frac{1}{2}} v^{-2}$	$u^{-\frac{1}{2}} v^{-2}$
$F(\partial_v, e_{\mathbb{S}^2})$, α, ϕ_1, ϕ_0	$u^{-\frac{1}{2}} v^{-3}$	$r^{-\frac{5}{2}}$	$r^{-\frac{5}{2}}$

One can also derive these components to complete the peeling result. The result is still the same as the one as CK.

Asymptotics for the spin-2 field

- In the interior region, for all weight σ : $|\phi_{ABCD}| \lesssim \frac{1}{<t>^\sigma}$.
- For the weight $\sigma = -\frac{11}{2}$, the exterior decay result is :

$$|\phi_{\mathbf{i}}| \lesssim \frac{1}{< v >^{1+4-\mathbf{i}} < u >^{\frac{1}{2}+\mathbf{i}}}.$$

- for the weight $\sigma = -\frac{7}{2}$:
  - for $\mathbf{i} = 2, 3, 4$,

$$|\phi_{\mathbf{i}}| \lesssim \frac{1}{< v >^{1+4-\mathbf{i}} < u >^{\frac{5}{2}+\mathbf{i}}}.$$

  - for $\mathbf{i} = 0, 1$,

$$|\phi_{\mathbf{i}}| \lesssim < r >^{-\frac{7}{2}}$$

  - Exactly the same result as CK.

Conclusion / Perspectives

- We recover fully Christodoulou-Klainerman results ; works for arbitrary spin, arbitrary weight.
- Purpose : extend this to Kerr space time.
- In this context, Maxwell fields of the form $\mathrm{d}\delta G$ are not charged.
- Make sense in this context to :

$$F = F_{Coulomb} + \mathrm{d}\delta G$$

Conclusion / Perspectives

- Hard : Need a proper elliptic theory, results on the wave equation are partially complete.
- Ideally : Hodge-Helmholtz-Kodaira decomposition on non compact manifolds with boundary for weighted Sobolev spaces.
- Also : Reduce the potential to one scalar potential (Debye potential) to use result for the scalar wave equation.
- There exists another process : spin lowering, which can generate both symmetries amongst solutions and potentials, using the existence of the Killing spinor.