The vector fields method for Vlasov fields

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Introduction

- (M, g) : a Lorentzian manifold ;
- Einstein equations in presence of matter : under the right gauge, system of quasilinear wave equations, coupled the equation governing the matter ;
- Geometric context : M space-like compact manifold $\rightarrow$ cosmology ; M asymptotically flat $\rightarrow$ isolated systems ;
- Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$: unique asymptotically flat, static solutions to the vacuum Einstein equations $\rightarrow$ ground state.
- Question : Stability of this ground state in the context of the Cauchy problem.

Introduction

- Stability of Minkowski space-time : proved by Christodoulou-Klainerman in 93, then Lindblad-Rodnianski in 2010.
- Particularity of the Lindblad-Rodnianski proof : "perturbative" proof, based solely on commutators with the flat wave equation ;
- Handle vacuum, matters modelled by a scalar field, or a Maxwell field...
- What about other matter fields ?

Introduction

- This talk : Vlasov matter : collisionless matter modelling the motion of a cloud of gas, the motion of galaxy clusters ;
- Mathematical model : system of quasilinear wave equations coupled with a transport equation ;
- Stability of Minkowski space-time proved in spherical symmetry for massive (Rein-Rendall 93), and massless particles (Dafermos 05) ;
- Announced in May '15 : Stability of Minkowski space-time as a solution to the Einstein equations coupled to the massless Vlasov matter, By M. Taylor ;
- Our aim : developing tools to handle perturbatively this problem.

Introduction

Tangent bundle

- (M, g) a smooth orientable, time orientable Lorentzian 4-manifold.
- Consider TM its tangent bundle, manifold of dimension 8, endowed with a metric of signature $(6, 2)$.
- Question : how do symmetries of the base manifold lift up into symmetries of the total manifold ?
- Huge tribute to pay to the very nice work of Sarbach and Zannias ('11, '13).

Complete lift

- Consider a vector field X on M.
- $p \in M$; γ be an integral curve of X with the initial data :

$$\begin{cases} \gamma(0) & = & p \\ \frac{d\gamma}{ds}(s) & = & X(\gamma(s)). \end{cases}$$

- Y defined on γ by Lie transporting the vector $X(p)$ along γ :

$$\mathcal{L}_X Y = [X, Y] = 0.$$

- Curve $\Gamma = (\gamma, Y(\gamma))$ on TM. The mapping

$$\tilde{X} : \begin{array}{ccc} TM & \longrightarrow & TTM \\ X(p) & \longmapsto & \frac{d\Gamma}{ds}(0) \end{array}$$

defines a vector field on TM, defined as being the complete field of X of M.

Complete lift

- Consider ϕ^t be a family of diffeomorphisms of M.
- On TM,

$$\phi^t_* = \begin{cases} TM & \to & TM \\ (x,v) & \mapsto & (\phi^t(x), d\phi^t_x(v)). \end{cases}$$

- X a vector field on M arising from ϕ^t :

$$X(x) = \frac{d\phi^t(x)}{dt}.$$

- Complete lift $\tilde{X}$:

$$\tilde{X}(x,v) = \frac{d\phi^t_*(x,v)}{dt}.$$

Using coordinates

- (U, x^α) a chart on M.
- $(TU, x^\alpha, v^\alpha = dx^\alpha)$ is a chart on TM, and provides a local trivialization of the bundle.
- In these coordinates, if $X = X^\alpha \frac{\partial}{\partial x^\alpha}$,

$$\tilde{X} = X^\alpha \frac{\partial}{\partial x^\alpha} + v^\alpha \frac{\partial X^\beta}{\partial x^\alpha} \frac{\partial}{\partial v^\beta} \quad (1)$$

$$= X^\alpha \underbrace{\left(\frac{\partial}{\partial x^\alpha} - v^\delta \Gamma^\beta_{\alpha\delta} \frac{\partial}{\partial v^\beta} \right)}_{=e_\alpha} + v^\alpha \nabla_\alpha X^\beta \frac{\partial}{\partial v^\beta} \quad (2)$$

$$= X^\alpha e_\alpha + v^\alpha \nabla_\alpha X^\beta \frac{\partial}{\partial v^\beta}. \quad (3)$$

Conformal Killing of the tangent bundle

- Example of complete lifts :

$$\frac{\partial}{\partial x^\alpha} \rightleftharpoons \frac{\partial}{\partial x^\alpha}$$

$$x_\beta \frac{\partial}{\partial x^\alpha} - x_\alpha \frac{\partial}{\partial x^\beta} \rightleftharpoons x_\beta \frac{\partial}{\partial x^\alpha} - x_\alpha \frac{\partial}{\partial x^\beta} + v_\beta \frac{\partial}{\partial v^\alpha} - v_\alpha \frac{\partial}{\partial v^\beta}$$

Proposition (Sasaki)

If X is Killing field, then $\tilde{X}$ is a Killing field for the Sasaki metric.

Rem : Not true for conformal Killing fields. Extra (sufficient) condition :

$$\nabla_\alpha \pi^{(X)}_{\beta\gamma} = 0$$

.

Mass shell

- M is time oriented.
- Define the submanifold Γ_m of TM :

$$\Gamma_m = \{X \in TM \,|\, g(X,X) = -m^2, \, X \text{ future pointing.}\}$$

Lemma

Let X be a vector field on M. Then its complete lift is tangent to the mass shell iff

- *if $m > 0$, X is Killing.*
- *if $m = 0$, X is conformal Killing.*

Geometry of the mass shell

- In the coordinates (x^α, v^α), the normal to the mass shell

$$N = v^\alpha \frac{\partial}{\partial v^\alpha}.$$

- X a vector field, $\tilde{X}$ its complete lift

$$g_s(N, \tilde{X}) = \pi^{(X)}_{\alpha\beta} v^\alpha v^\beta.$$

Vlasov field

- Vlasov matter : collisionless matter, made of particle of a given mass m.
- $f : \Gamma_m \to \mathbb{R}$: density of particles at one point on the mass shell (position+ speed).
- These particles follow geodesics.
- Equations for f given by the geodesics spray T, which is tangent to the mass shell Γ_m :

$$T = v^\alpha e_\alpha = v^\alpha \frac{\partial}{\partial x^\alpha} - v^\mu v^\nu \Gamma^\alpha_{\mu\nu} \frac{\partial}{\partial v^\alpha}.$$

- f satisfies

$$T(f) = 0.$$

Commutators with the Vlasov equation

- X a vector field on M, $\tilde{X}$ its complete lift.

Lemma

$$[T, \tilde{X}] = v^\alpha v^\beta \left[\nabla_\alpha \nabla_\beta X^\mu - R^\mu{}_{\beta\alpha\nu} X^\nu\right] \frac{\partial}{\partial v^\mu}.$$

- On Minkowski space-time, generators of isometries :

$$\partial_{x^\alpha}, x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha}$$

- If $m > 0$, the commutators are :

$$\partial_{x^\alpha}, x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha} + v_\alpha \partial_{v^\beta} - v_\beta \partial_{v^\alpha}$$

- If $m = 0$, add the complete lift of the scaling $x^\alpha \partial_{x^\alpha}$:

$$x^\alpha \partial_{x^\alpha} + v^\alpha \partial_{v^\alpha}$$

Setup

- Consider Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^n_x$
- Corresponding phase space : $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$.
- Transport operator : if $f : \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v \to \mathbb{R}$,

$$Tf = v^\alpha \partial_{x^\alpha} f = 0 \text{ with } v^0 = \sqrt{m^2 + \sum_i (v^i)^2}.$$

- If f is a solution, $T|f| = 0$ is a solution.
- Conserved quantity :

$$\int_{\{t\}\times\mathbb{R}^n} \underbrace{\int_v |f| dv}_{=\rho(|f|)} dx.$$

- Question : What can we say about the decay of $\rho(|f|)$?

Integration on the fiber of the mass shell

- X vector field, whose complete lift is tangent to the mass shell, $f : \Gamma_m \to \mathbb{R}$.

Lemma

$$\mathcal{L}_X \left(\int_{\Gamma_m(x)} f d\mu_{\Gamma_m(x)} \right) = \int_{\Gamma_m(x)} \mathcal{L}_{\tilde{X}} (f)\, d\mu_{\Gamma_m(x)} - \frac{1}{2} \int_{\Gamma_m(x)} f g^{\alpha\beta} \pi^{(X)}_{\alpha\beta} d\mu_{\Gamma_m(x)}.$$

Case $m = 0$

- Approach widely inspired by the work of Klainerman '85, '89.
- Set of commutators $\hat{K}$:

$$\partial_{x^\alpha}, x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha} + v_\alpha \partial_{v^\beta} - v_\beta \partial_{v^\alpha}, x^\alpha \partial_{x^\alpha} + v^\alpha \partial_{v^\alpha}.$$

- if $\hat{Z}_1, \ldots, \hat{Z}_k \in \hat{\mathbb{K}}$,

$$\int_x |Z^1 \ldots Z^k \rho(f)| dx \leq \int_v \rho(|\hat{Z}^1 \ldots \hat{Z}^k f|) dx = \text{ constant } .$$

- We control weighted L^1-norm of f.
- Right tool : adapted Klainerman-Sobolev estimates.

Decay for massless fields

Defined $E_n^{\mathbb{K}}[f](t) = \sum_{k=0}^n \sum_{\hat{Z}_1,\ldots,\hat{Z}_k \in \hat{\mathbb{K}}} \int_{\{t\}} \rho(|Z^1 \ldots Z^k f|) dx$.

Proposition (Adapted Klainerman-Sobolev estimates)

For any f, for all $t \geq 0, x \in \mathbb{R}^n$,

$$(1 + |t - x|)(1 + t + |x|)^{n-1} \rho(|f|) \lesssim E_n^{\mathbb{K}}[f](t)$$

Theorem (Decay for massless fields)

If initially $E_n^{\mathbb{K}}(f)(t = 0) < \infty$, then

$$(1 + |t - x|)(1 + t + x)^{n-1} \rho(|f|) \lesssim E_n^{\mathbb{K}}[f](t = 0)$$

Case $m > 0$

- Same set up as previously.
- Set of commutators $\hat{\mathbb{P}}$.

$$\partial_{x^\alpha}, x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha} + v_\alpha \partial_{v^\beta} - v_\beta \partial_{v^\alpha},$$

- Mimick Klainerman's approach of Klein-Gordon ('93) : work on hyperboloids.
- Introduce

$$T^2 = t^2 - r^2.$$

Decay for massive fields

Defined
$E_n^{\mathbb{P}}[f](T) = \sum_{k=0}^{n} \sum_{\hat{Z}_1,\dots,\hat{P}_k \in \hat{\mathbb{P}}} \int_{H_T} \rho\left(\frac{tv^0 - x_i v^i}{v^0} |\hat{Z}^1 \dots \hat{Z}^k f| \right) dx.$

Proposition (Adapted Klainerman-Sobolev estimates)

For any f, for all $T \geq 1, x \in \mathbb{R}^n$,

$$t^n \rho(|f|(v^0)^{-1}) \lesssim E_n^{\mathbb{P}}[f](T)$$

Theorem (Decay for massless fields)

If initially $E_n^{\mathbb{P}}[f](T=1) < \infty$, then

$$t^n \rho(|f|(v^0)^{-1}) \lesssim E_n^{\mathbb{P}}[f](T=1)$$

The Vlasov Nordström system

- Consider $\hat{g} = e^{2\phi}\eta$ a conformally flat metric on $\mathbb{R}_t \times \mathbb{R}^n_x$, and consider the associated geodesic spray $\hat{T}$. Let $\hat{f}$ be a solution of the transport equation, on the mass shell of mass $m \geq 0$. :

$$\hat{T}\hat{f} = 0.$$

- After rescaling $\hat{f}$, and the phase space variable, take f

$$Tf + T(\phi)v^i\partial_{v^i}f = 0 \text{ if } m = 0$$

and

$$Tf - (T(\phi)v^i + \nabla^i\phi)v^i\partial_{v^i}f = (n+1)T(\phi)f \text{ if } m > 0$$

- Coupled with $\Box\phi = -m^2\int_v \frac{f}{v^0}dv$.
- Global-well posedness : Calogero-Rein '02-'04.
- Stability ?

The massless case

- Consider the system

$$\begin{cases} T_\phi f = Tf + T(\phi) v^i \partial_{v^i} f = 0 \\ \Box \phi = 0 \text{ or } = Q(\partial\phi, \partial\phi) \end{cases}$$

- Set of commutators : $A = \widehat{\mathbb{K}} \cup \{v^i \partial_{v^i}\}$.
- Conserved quantity :

$$\int_{\{t\}\times\mathbb{R}^n_x} \int_v v^0 f dv dx \approx \text{constant} \ ,$$

- Define :

$$E_N[f](t) = \sum_{|\alpha|\leq N} \sum_{\hat{Z}\in A} \int_x \int_v v^0 |\hat{Z}^\alpha f| dv dx.$$

- Proof : Direct application of Gronwall.

Decay for massless Vlasov-Nordström fields

Theorem

Let $N \geq n + \lfloor n/2 \rfloor + 2$*. Assume inititally*

$$\mathcal{E}_N[\phi](t=0) < \infty \text{ and } E_N[f](t)(t=0) < \infty.$$

Then,

- *for* $n \geq 4$*, and* $N' < N - n$*,*

$$\rho(|\widehat{Z}^{N'} f|) \lesssim \frac{E_N[f](t=0) e^{C E_N[\phi](t=0)}}{(1+t+r)^{n-1}(1+|t-r|)}.$$

Improvement for $n = 3$

- New set of commutators : take $\widehat{\mathbb{K}} \cup \{v^i \partial_{v^i}\}$; add the weights

$$k = \{x^\alpha v_\alpha, v^\alpha x^\beta - x^\alpha v^\beta, v^\alpha\}$$

- if $\mathfrak{z} \in k$, $T\mathfrak{z} = 0$.
- Key point : improved decay for $T\phi$:

$$|T\phi| \leq v^0 u^{-\frac{1}{2}} v^{-1}$$

- Write

$$|T\phi| = |v^0 \left(\partial_t + \frac{x^i}{r}\frac{\partial}{\partial x^i}\right)\phi - \frac{v^0}{t}\frac{x^i}{r}\left(\frac{-x^j\Omega_{ij} + t\Omega_{0i} - x^i S}{t+r}\right)\phi$$
$$+ \frac{v^i t - x^i v^0}{t}\frac{\partial}{\partial x^i}\phi| \leq \mathfrak{z} u^{-\frac{1}{2}} v^{-1} t^{-1}.$$

Decay for $n = 3$

Introduce the energy

$$E_{N,q}[f] \equiv \sum_{|\alpha| \leq N} \iint_{x,v} v^0 |\mathfrak{z}^q \hat{Z}^\alpha f| dv dx.$$

Theorem

- *if $N > 5$, and $N' < N - 3$, $q > 0$ then*

$$\rho(|\widehat{Z}^{N'} f|) \lesssim E_{N',q}(0) \frac{t^{C\mathcal{E}_{N'}(\phi)(0)}}{(1+t+r)^2(1+|t-r|)}.$$

- *if $N > 6$, and $N' < N - 4$, $q > 0$ then*

$$\rho(|\widehat{Z}^{N'} f|) \lesssim E_{N',q+1}[f](0) e^{C\mathcal{E}_N[\phi](0)} (1+t+r)^{-2} (1+|t-r|)^{-1}.$$

The massive Vlasov Nordström system

- Consider the system

$$\begin{cases} T_\phi f = Tf - (T(\phi)v^i + \nabla^i \phi) v^i \partial_{v^i} f - (n+1) T(\phi) f = 0 \\ \Box \phi = \int_v \frac{f}{v^0} dv \ \text{ possibly } \ + Q(\partial\phi, \partial\phi) \end{cases}$$

- Model for conformally flat perturbations of Einstein equations coupled with massive Vlasov matter,
- Spatial dimension $n > 3$.
- List of problems
  - Different sets of commutators
  - Real coupling (though only linear)
  - Main technical difficulty : establish L^2-estimates when a high number of derivatives hits ϕ and f simultaneously

The massive Vlasov Nordström system for $n > 3$

- Different sets of commutators : solved by working on hyperboloids ;
- Coupling : handled by the commutators.
- Techniques : bootstrap argument requiring a renormalization of the system for the transport part.

L^2-estimates

- Establishing decay for the L^2-norms for $\rho(\widehat{Z}^\alpha f)$ when α is big
- Check that $\Psi = Z^\alpha f$ satisfies an equation of the form

$$T(\Psi) + (\text{good perturbations})\,\Psi = hg$$

  where h is $L^2_x L^1_v$-uniformly bounded, and g satisfies $T(g) = 0$ + Klainerman-Sobolev type inequality ;
- homogeneous problem dealt with by assuming higher regularity on the initial data + commutations ;

L^2-estimates

- Inhomogeneous problem + zero ID : $\Psi = gH$ with $T(H) = h$:

$$T(\Psi) = T(g)H + T(H)g = hg.$$

- if $\Psi = gH$, then

$$||\psi||_{L^2_x L^1_v} \leq ||g||^{1/2}_{L^\infty_x L^1_v} ||gH^2||^{1/2}_{L^1_x L^1_v}$$

- gH^2 satisfies a good perturbed transport equation.
- Finally :

$$||\psi||_{L^2_x L^1_v} \leq \underbrace{||g||^{1/2}_{L^\infty_x L^1_v}}_{\text{decays fast}} \underbrace{||gH^2||^{1/2}_{L^1_x L^1_v}}_{\text{conserved}}$$

Pointwise decay for solutions of the Vlasov-Nordstrom system in dimension $n > 3$.

Theorem : Global existence - asymptotic behavior for small data (FJS - 2015)

Let $N > 3n+4$; Assume that the initial data on H_1 are small enough

$$E_{N+n}[f](1) + \mathcal{E}_N[\phi](1) \leq \epsilon;$$

- Then, for all $T > 1$, $E_N[f](T) \lesssim \epsilon T^{C\epsilon^{1/4}}$ and $\mathcal{E}_N[\phi](T) \lesssim \epsilon$;
- Furthermore,

$$\int_{H_T} \frac{t}{\rho} \left(\int_v \widehat{Z}^N f \frac{dv}{v^0} \right)^2 d\mu_{H_\rho} \lesssim \epsilon^2 T^{C\epsilon^{1/4}-n}$$

The case n=3

- Requires better decay for the wave to deal with the transport equation ;
- Solved by resorting to more general multiplier estimates, or better commutators ;
- Require better integrability properties on hyperboloids ;
- Solved by resorting to the better decay in outgoing directions.
- Ongoing work...

Conclusion

- Vector fields method which works for Vlasov fields.
- Multipliers estimates to prove integrated decay also work fine.
- This method of commutators also works for classical Vlasov !
- Good tool to couple with gravity. In particular, one could thing at applying a pure vector field method used to handle Einstein equations by Lindblad and Rodnianski.
- Big questions : global existence for small data ? Stability of Minkowski, as a solution to the Einstein-Vlasov system ?