An integrated decay estimate for massless Vlasov fields

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Introduction

- Use of symmetries to study Vlasov fields introduced by FJS15 for the massive/massless Vlasov fields.
- Purpose : Develop a perturbative approach of the stability of Minkowski spacetime, as a solution to the Einstein - massive Vlasov system, similar to the Lindblad-Rodniandki proof.
- But offer also the possiblity to extend the study of Vlasov fields on other given backgrounds, for instance black hole backgrounds.
- Existence of steady states for massive particles on Kerr : Sarbach-Rioseco.
- Steady states at "high energy" with BH for massless particles when coupled to gravity : suggested by the work of Andreasson-Fajman-Thaller.

Introduction

- But dispersion is expected for purely massless fields, because of the instability of trapped null geodesics.
- Furthermore, duality Vlasov / wave : massless Vlasov fields offer a good playground to experiment and develop new methods for the wave equation.
- This work : illustrates this duality between wave / massless fields ;
- Adapt to massless Vlasov fields the method based on hidden symmetries developed by Andersson-Blue.
- Core of this work : Andersson-Blue paper proving pointwise estimates for the wave equation mixed with symmetry approach by FJS.
- Prove existence of a bounded conserved energy and an integrated decay estimate.

Organization of the talk

Symmetries for Vlasov fields

Defining the energy

Morawetz estimate

Setup

- Purpose here : Compare framework in FJS with a standard Hamiltonian theory of commutators.
- Setup : (M, g) Lorentzian manifold (oriented, time oriented). Consider $T^\star M$.
- Chart (U, q^α) on M ; natural extension to $T^\star U$: $(U, q^\alpha, p_\alpha = \partial_{q^\alpha})$.
- On $T^\star M$, canonical non-degenerate (symplectic) 2-form :

$$\epsilon = dq^\alpha \wedge dp_\alpha.$$

- Functions on $T^\star M$: Hamiltonian.
- Representations of dH :

$$dH(V) = \epsilon(X_H, V), \quad X_H = \frac{\partial H}{\partial p_\alpha}\frac{\partial}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha}\frac{\partial}{\partial p_\alpha}$$

Zero-th order symmetry operator

- Example : Geodesic spray / Liouville vector field :

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, \quad \mathcal{X} := X_H = g^{\alpha\gamma} p_\alpha \frac{\partial}{\partial q^\gamma} - \frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial q^\gamma} p_\alpha p_\beta \frac{\partial}{\partial p_\gamma}$$

- If X is a vector field, $K_X = X^\alpha p_\alpha$
- Obvious fact :

$$\mathcal{X}(K_X) = p_\alpha p_\beta \left(\pi^{(X)} \right)^{\alpha\beta}.$$

- Hence, zero-th order symmetry operator of $\mathcal{X}$: if X Killing,

$$\text{If } \mathcal{X}(f) = 0 \text{ then } \mathcal{X}(K_X \cdot f) = 0.$$

First order symmetry operator

- Standard general fact : Taking two Hamiltonians H, K :

$$[X_K, X_H] = 0 \text{ iff } \{K, H\} = \frac{\partial H}{\partial p_\alpha}\frac{\partial K}{\partial q^\alpha} - \frac{\partial H}{\partial q^\alpha}\frac{\partial K}{\partial p_\alpha} = 0$$

- If X is a vector field, $K = X^\alpha p_\alpha$
- Associated Hamiltonian vector field :

$$X_K = X^\alpha \frac{\partial}{\partial q^\alpha} - p_\alpha \frac{\partial X^\alpha}{\partial q^\gamma}\frac{\partial}{\partial p_\gamma}$$

- General formula :

$$[\mathcal{X}, X_K] = -2\pi^{(K)\alpha\gamma} p_\alpha \frac{\partial}{\partial q^\gamma} + \left(\partial_{q^\gamma}\pi^{(K)}\right)^{\alpha\beta} p_\alpha p_\beta \frac{\partial}{\partial p_\gamma}$$

- In particular, if X is Killing

$$\text{If } \mathcal{X}(f) = 0 \text{ then } \mathcal{X}(X_K \cdot f) = 0.$$

Comparison with FJS

- On TM, use the mapping between $T^\star M$ and TM :

$$x^\alpha = q^\alpha \text{ and } v^\alpha = g^{\alpha\beta} p_\beta.$$

- The image of X_K writes

$$X_K = X^\alpha e_\alpha - v_\alpha \nabla^\gamma X^\alpha \frac{\partial}{\partial v^\gamma}$$

- The image of the commutator :

$$[\mathcal{X}, X_K] = -2\pi^{(X)\alpha\gamma} v_\gamma e_\alpha + \nabla^\gamma \pi^{(X)}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\gamma}$$

- Notion used in FJS : complete lift

$$\tilde{X} = X^\alpha e_\alpha - v^\alpha \nabla_\alpha X^\gamma \frac{\partial}{\partial v^\gamma}$$

Complete lift

- Consider ϕ^t be a family of diffeomorphisms of M.
- On TM,

$$\phi^t_* = \begin{cases} TM & \to & TM \\ (x,v) & \mapsto & (\phi^t(x), d\phi^t_x(v)). \end{cases}$$

- X a vector field on M arising from ϕ^t :

$$X(x) = \frac{d\phi^t(x)}{dt}.$$

- Complete lift $\tilde{X}$:

$$\tilde{X}(x,v) = \frac{d\phi^t_*(x,v)}{dt}.$$

Comparison with FJS 2

- When X is Killing, $\tilde{X} = X_K$.
- But, perturbatively

$$[\mathcal{X}, X_K] = -2\pi^{(X)\alpha\gamma} v_\gamma e_\alpha + \nabla^\gamma \pi^{(X)}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\gamma}$$
$$[\mathcal{X}, \tilde{X}] = v^\alpha v^\beta \left[\nabla_\alpha \nabla_\beta X^\mu - R^\mu{}_{\beta\alpha\nu} X^\nu\right] \frac{\partial}{\partial v^\mu}.$$

- Both coincides because Killing fields are Jacobi fields along geodesics.
- Which one is better perturbatively ?

Higher order symmetry operators

- If Q is a symmetric 2-tensor, $K = \frac{1}{2} Q^{\alpha\beta} p_\alpha p_\beta$
- Immediate calculation

$$\mathcal{X}\left(Q^{\alpha\beta} p_\alpha p_\beta\right) = \nabla^{(\alpha} Q^{\beta\gamma)} p_\alpha p_\beta p_\gamma$$

- If Q is Killing,

$$\text{If } \mathcal{X}(f) = 0 \text{ then } \mathcal{X}(Q^{\alpha\beta} p_\alpha p_\beta \cdot f) = 0.$$

- Similarly

$$X_K = Q^{\alpha\beta} p_\alpha \frac{\partial}{\partial q^\alpha} - \frac{1}{2} p_\alpha p_\beta \frac{\partial Q^{\alpha\beta}}{\partial q^\gamma} \frac{\partial}{\partial p_\gamma}$$

- On Minkowski : $Q^{\alpha\beta} = X_1{}^{(\alpha} X_2{}^{\beta)}$, with X_1, X_2 Killing.
- Not true on Kerr : Carter constant.

Kerr spacetime

- Slowly rotating black holes ($a << M$) : stationary, axisymmetric solution to the EE in vacuum. In Boyer-Lindquist coordinates, on $M = \mathbb{R} \times [r_+, +\infty) \times \mathbb{S}^2$

$$g = -\left(1 - \frac{2Mr}{\Sigma}\right) \mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \frac{\Pi \sin^2\theta}{\Sigma}\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2,$$

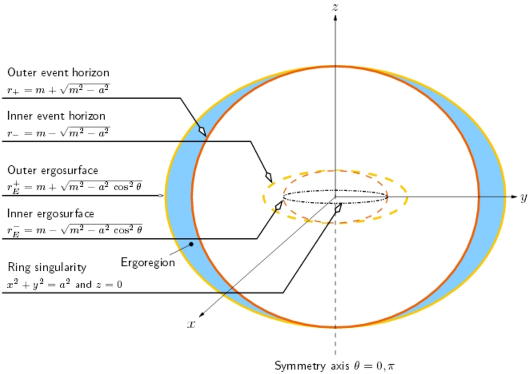
where $r_{outer} = M + \sqrt{M^2 - a^2}$, and

$$\Delta = r^2 - 2Mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$
$$\Pi = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta.$$

- Killing vectors : $\partial_t, \partial_\phi$, a non trivial Killing 2-tensor :

$$Q^{ab} = \partial_\theta^a \partial_\theta^b + \cot^2\theta \partial_\phi^a \partial_\phi^b + a^2 \sin^2\theta \partial_t^a \partial_t^b$$

Diagram



∂_t is not timelike in the blue region (ergoregion).

Andersson-Blue work on the wave equation

- Difficulty : less symmetries, ∂_t not positive, i.e. no positive energies out of ∂_t.
- Work of Andersson-Blue : define a positive energy, reinforced but the use of symmetries ; based on the analysis of geodesics, define the appropriate vector fields to perform Morawetz estimates ; prove the positive definite energy is approximatively conserved based on this estimates ; then commute to get pointwise decay estimates.
- One obstruction : the geodesics analysis is true only at high frequency for the wave equations.
- for massless Vlasov fields, no problem.

The setting

- Mass shell = bundle of future oriented light cone : $\mathcal{C}^+$.
- Massless Vlasov equation : $f : \mathcal{C}^+ \to \mathbb{R}^+$

$$\mathcal{X} f = v^a \left(\frac{\partial}{\partial x^a} - v^b {\Gamma^c}_{ab} \frac{\partial}{\partial v^c} \right) f.$$

- Zero-th order symmetries

$$e = v_a \partial_t^a, l_z = v_a \partial_\phi^a, q = v_a v_b Q^{ab}$$
$$\mathbb{S}_2 = \{e^2, e l_z, l_z^2, q\} = \{S_{\underline{a}}\}_{\underline{a}}.$$

- Task : define a positive energy, prove its conservation, establish a morawetz estimates, and prove an integral energy decay.

The volume form

- Usual form : at $x = (x^0, \dots, x^3)$, ∂_{x^0} timelike

$$T_{00} = \rho(f)(x) = \int_{\mathcal{C}_x^+} f v_t^2 \sqrt{|g|} \frac{dv^1 dv^2 dv^3}{|v_t|}$$

- Problem : here ∂_t is not uniformly timelike.
- Solution : Use the Gelfand-Leray form :

$$\mathcal{C}_x^+ = \{S(v) = g_x(v,v); v \text{ future oriented}\} \subset T_x M;$$

factorize the metric w.r.t. $d_v S$:

$$\sqrt{|g|} dv^0 \wedge \dots \wedge dv^3 = d_v S \wedge d\mu_{\mathcal{C}_x^+} = d_v S \wedge \left(\sqrt{|g|} \frac{dv^1 \wedge dv^2 \wedge dv^3}{-v_t} \right)$$

- Orienting $\mathcal{C}_x^+$ using the orientation/ time orientation of the manifold, one obtains : if f is positive :

$$\int_{\underbrace{\mathcal{C}_x^+}_{\text{oriented}}} \underbrace{f \sqrt{|g|} \frac{dv^1 \wedge dv^2 \wedge dv^3}{-v_t}}_{\text{form}} = \int_{\underbrace{\mathcal{C}_x^+}_{\text{measurable set}}} \underbrace{f \sqrt{|g|} \frac{dv^1 dv^2 dv^3}{|v_t|}}_{\text{measure}} \geq 0$$

Strengthening the energy

- Exploit the symmetries of order 2 : remind $\mathcal{X}\left(S_{\underline{a}_1} S_{\underline{a}_2} f\right) = 0$:

$$\mathrm{T}_{\alpha\beta}[S_{\underline{a}_1} S_{\underline{a}_2} f] = \int_{\mathcal{C}_x^+} \left(S_{\underline{a}_1} S_{\underline{a}_2} f\right) v_\alpha v_\beta d\mu_{\mathcal{C}_x^+}.$$

- Choose a vector $X^{a\underline{a}_1\underline{a}_2}$ wisely : depends on the symmetries it applies to !
- Define the energy : on a spacelike Cauchy hypersurface Σ_t :

$$E_X[f](\Sigma_t) = \int_{\Sigma_t} \mathrm{T}_{ab\underline{a}_1\ldots\underline{a}_k}[f] X^{a\underline{a}_1\cdots\underline{a}_k} \mathrm{d}\nu^b_{\Sigma_t}$$

- Difference between two slices :

$$E_X[f](\Sigma_{t_2}) - E_X[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{X,\Omega,q}[f](R)\mathrm{d}\mu_g, \text{ where}$$

$$\Pi_{X,\Omega,q}[f] = -\frac{1}{2}\Omega^2 \mathrm{T}_{ab\underline{a}_1\ldots\underline{a}_k}[f]\mathrm{Lie}_{X^{\underline{a}_1\cdots\underline{a}_k}}(\Omega^{-2}g^{ab}) + \mathrm{T}_{ab\underline{a}_1\ldots\underline{a}_k} g^{ab}[f] q^{\underline{a}_1\cdots\underline{a}_k}.$$

Principle

- Principle of the estimates : find $\mathbf{T}_\chi$ and $\mathbf{A}$, such that

$$E_{\mathbf{T}_\chi} \geq 0, \leftarrow \text{choose } \mathbf{T}_\chi \text{timelike and causal}$$

$$\Pi_{\mathbf{A}} \geq 0, \leftarrow \text{choose } q \text{ and } \Omega \text{ wisely and take}$$

$$\Omega^{-2}\Pi_{\mathbf{A},\Omega,q} = \left(-\frac{1}{2}\mathrm{Lie}_{\mathbf{A}}(\Omega^{-2}g^{ab}) - q\Omega^{-2}g^{ab}\right)\mathrm{T}_{ab}$$

$$\Pi_{\mathbf{T}_\chi} \lesssim \underbrace{\frac{|a|}{M}}_{\approx|\mathrm{Lie}_{\mathbf{T}_\chi}(g^{ab})|} \Pi_{\mathbf{A}}$$

$$E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}| \text{ provided that } |\mathbf{A}| \lesssim |\mathbf{T}_\chi|$$

Defining the model energy 1

First step : Choosing the vector field :

Definition

$$T_\perp = \left(\partial_t + \frac{2aMr}{\Pi}\partial_\phi\right)^a = (\partial_t + \omega_\perp \partial_\phi)^a\,,$$

$$T_\chi^a = (\partial_t + \chi\omega_{\mathcal{H}}\partial_\phi)^a, \leftarrow \textit{uniformly timelike}$$

$$\mathbf{T}_\chi^{a\underline{ab}} = T_\chi^a \delta^{\underline{ab}} \leftarrow \textit{symmetry strengthened VF},$$

where $\omega_{\mathcal{H}} = a/(r_+^2 + a^2)$ *is the rotation speed of the horizon,* $\chi = \chi(r)$ *is a function that is* 1 *for* $r < r_\chi$*, smoothly decreasing on* $r \in [r_\chi, r_\chi + M]$*, and identically* 0 *for* $r > r_\chi + M$*, and where* r_χ *is chosen sufficiently large. For simplicity, we take* $r_\chi = 10M$*.*

Defining the model energy 2

Lemma

There is a positive constant $\bar{\epsilon}$ such that if $|a| < \bar{\epsilon} M$, $t \in \mathbb{R}$, and $f : \mathcal{C}^+ \to [0, \infty)$ is continuous, then

$$
\begin{aligned}
& E_{T_\perp}[f](\Sigma_t) \\
& \quad \simeq \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(\frac{(r^2 + a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) f \mathrm{d}\mu_{\mathcal{C}_x^+} \mathrm{d}\mu_{\Sigma_t}, \\
E_{T_\perp}[f](\Sigma_t) \simeq & E_{T_\chi}[f](\Sigma_t) \\
\simeq & E_{model,3}[f](t).
\end{aligned}
$$

Proof : Compare wisely polynomials in v's. Exploit some uniformity in the dependency in a/M.

The Morawetz estimate : The starting point

- Key point : Understand how the instability of trapped (orbiting) null geodesics is encoded in the metric coefficients.
- Write

$$\Sigma g^{ab} = \Delta \partial_r^a \partial_r^b + \frac{1}{\Delta} \mathcal{R}^{ab},$$

where

$$\begin{aligned}
\Delta &= r^2 - 2Mr + a^2, \\
\Sigma &= \Omega^{-2} = r^2 + a^2 \cos^2 \theta, \\
\mathcal{R}^{ab} &= -(r^2 + a^2)^2 \partial_t^a \partial_t^b - 4aMr \partial_t^{(a} \partial_\phi^{b)} + (\Delta - a^2) \partial_\phi^a \partial_\phi^b + \Delta Q^{ab}, \\
Q^{ab} &= \partial_\theta^a \partial_\theta^b + \cot^2 \theta \partial_\phi^a \partial_\phi^b + a^2 \sin^2 \theta \partial_t^a \partial_t^b.
\end{aligned}$$

Morawetz estimate : geodesic equation

- Stolen from unpublished notes of Pieter ;
- γ a null geodesic, with conserved quantities :

$$e = -\dot{\gamma}_t,$$
$$l_z = -\dot{\gamma}_\phi,$$
$$q = \dot{\gamma}_\theta^2 + \frac{\cos^2\theta}{\sin^2\theta}\dot{\gamma}_\phi^2 + a^2\sin^2\theta\dot{\gamma}_t^2.$$

- Radial component :

$$\Sigma^2\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = -\mathcal{R}(r; M, a; e, l_z, q),$$

where

$$\mathcal{R}(r; M, a; e, l_z, q) = -(r^2+a^2)^2e^2 - 4aMrel_z + (\Delta - a^2)l_z^2 + \Delta q.$$

- Claim : for $a << M$, unstable geodesics : around $r \approx 3M$, $\partial_r\mathcal{R} = 0, \partial_r^2\mathcal{R} < 0$.

Choosing the Morawetz vector field

Definition

If z and w are smooth functions of r and the parameters M and a, the Morawetz vector field and the reduced scalar functions are defined to be

$$\mathbf{A}^{a\underline{ab}} = -zw\mathcal{L}^{(\underline{a}}\tilde{\mathcal{R}}'^{\underline{b})}\partial_r^a,$$
$$q^{\underline{ab}} = \frac{1}{2}(\partial_r z)w\mathcal{L}^{(\underline{a}}\tilde{\mathcal{R}}'^{\underline{b})},$$

where

$$\mathcal{R} = \mathcal{R}^{ab}v_a v_b = \mathcal{R}^{\underline{a}}S_{\underline{a}} = \mathcal{R}^{\underline{a}}S_{\underline{a}}^{ab}v_a v_b$$
$$\tilde{\mathcal{R}}'^{\underline{a}} = \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),$$
$$\mathcal{L} = \mathcal{L}^{\underline{a}}S_{\underline{a}} = M^2e^2 + l_z^2 + q,$$

Choosing the weights z and w

- z and w are chosen so that
    - comparability with $\Pi_{\mathbf{T}_\chi}$;
    - right combination of $\mathcal{R}$ and derivatives ;
    - compare with T_χ.
- The same weights as Andersson-Blue.
- Really technical and delicate issue, obtained by carefully comparing expressions of bulk terms.
- Choose $\Omega^{-2} = \Sigma$,

Flashing the weights z and w

Definition

Given a positive value for the parameter ϵ_{e^2}, we use the following weights to define the Morawetz vector field,

$$z = z_1 z_2, \qquad w = w_1 w_2,$$

$$z_1 = \frac{\Delta}{(r^2+a^2)^2}, \qquad w_1 = \frac{(r^2+a^2)^4}{3r^2-a^2},$$

$$z_2 = 1 - M^2 \epsilon_{e^2} \frac{\Delta}{(r^2+a^2)^2}, \qquad w_2 = \frac{1}{2r}.$$

Same weights as in Andersson-Blue $\leftarrow$ real technical difficulty.

Estimates on the Morawetz bulk term

Lemma

There are positive constants $\bar{\epsilon}$, ϵ_{e^2}, and C such that if $|a| \leq \bar{\epsilon}M$, $0 < \epsilon_{e^2} \leq \bar{\epsilon_{e^2}}$ and $f : \mathcal{C}^+ \to [0, \infty)$ is a solution of the Vlasov equation, then

$$C\Omega^2\Pi_{\mathbf{A}} \geq M\frac{\Delta^2}{(r^2+a^2)^2}v_r^2|f|_2 + r^5\tilde{\mathcal{R}}'\tilde{\mathcal{R}}'\mathcal{L}f.$$

and

$$\begin{aligned}\tilde{\mathcal{R}}' = &-2r^{-4}(r-3M)\mathcal{L}_{\epsilon_{e^2}}\\ &+aMO(r^{-4})el_z\\ &+a^2\left(O(r^{-5})q+O(r^{-5})l_z^2\right)\\ &+M^2\epsilon_{e^2}\left(a^2O(r^{-5})e^2+O(r^{-5})q+O(r^{-5})l_z^2\right).\end{aligned}$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{T}_\chi} \mathrm{d}\mu_g$$

$$E_{\mathbf{A}}[f](\Sigma_{t_2}) - E_{\mathbf{A}}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- the energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{T}_\chi} \mathrm{d}\mu_g$$

$$E_{\mathbf{A}}[f](\Sigma_{t_2}) - E_{\mathbf{A}}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

Closing argument

- Remind

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- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{|a|}{M}\Pi_{\mathbf{A}}\mathrm{d}\mu_g$$

$$E_{\mathbf{A}}[f](\Sigma_{t_2}) - E_{\mathbf{A}}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}}\mathrm{d}\mu_g$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{|a|}{M}\Pi_{\mathbf{A}}\mathrm{d}\mu_g$$

$$E_{\mathbf{A}}[f](\Sigma_{t_2}) - E_{\mathbf{A}}[f](\Sigma_{t_1}) = \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}}\mathrm{d}\mu_g$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \lesssim \int_{t_1}^{t_2}\int_{\Sigma_t} \frac{|a|}{M}\Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) + E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) - E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \lesssim \frac{|a|}{M}\left(E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) + E_{\mathbf{T}_\chi}[f](\Sigma_{t_1})\right)$$

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) + E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

Closing argument

- Remind

$$E_{\mathbf{T}_\chi} \geq 0,\ \Pi_{\mathbf{A}} \geq 0,\ \Pi_{\mathbf{T}_\chi} \lesssim \frac{|a|}{M}\Pi_{\mathbf{A}},\ E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|.$$

- Energy estimates

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) \lesssim \frac{1+\frac{|a|}{M}}{1-\frac{|a|}{M}} E_{\mathbf{T}_\chi}[f](\Sigma_{t_1})$$

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) + E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}) \gtrsim \int_{t_1}^{t_2}\int_{\Sigma_t} \Pi_{\mathbf{A}} \mathrm{d}\mu_g$$

- Finally use the estimates for $\Pi_{\mathbf{A}}$, and take $t_1 \to -\infty$ and $t_2 \to +\infty$.

Theorem 1 : Existence of a bounded energy

Theorem

There are positive constants C and $\bar{\epsilon}$ such that if $M > 0$, $|a| \leq \bar{\epsilon} M$, and $f : \mathcal{C}^+ \to [0, \infty)$ is a smooth solution of the Vlasov equation in the exterior of the Kerr spacetime with parameters (M, a), then, for all t in $\mathbb{R}$,

$$E_{model,3}[f](t) \leq C E_{model,3}[f](0).$$

Theorem 2 : Integrated decay estimates

Theorem (Part 1)

There are positive constants C, $\bar{\epsilon}$, and $\bar{r}$ and a function $\mathbf{1}_{r \not\approx 3M}$ which is identically 1 for $|r-3M| \geq \bar{r}$ and zero otherwise such that if $M>0$, $|a| \leq \bar{\epsilon} M$, and $f : \mathcal{C}^+ \to [0,\infty)$ is a smooth solution of the Vlasov equation in the exterior of the Kerr spacetime with parameters (M,a), then,

$$\int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(M \frac{\Delta^2}{(r^2+a^2)^2} \right) v_r^2 |f|_2 + \mathbf{1}_{r \not\approx 3M} \frac{1}{r} \left(M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2 \right) |f|_2 \mathrm{d}^3 v \mathrm{d}^4 x \leq C E_{model,3}[f](0)$$

where

$$|f|_2 = \left| M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2\theta} v_\phi^2 \right|^2 f$$

Theorem 2 : Integrated decay estimates

Theorem (Part 2)

More precisely,

$$\int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} M \frac{\Delta^2}{(r^2+a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \, \mathrm{d}\mu_{\mathcal{C}_x^+} \mathrm{d}\mu_g \leq C E_{model,3}[f](0),$$