Stability of Minkowski space as a solution to the Einstein-Vlasov system

Jérémie Joudioux

Gravitationsphysik
Fakultät für Physik
Universität Wien

Joint work with
D. Fajman (Uni. Wien)
J. Smulevici (Orsay & ENS)

FJS15 : Analysis & PDE 10-7, 1539–1612
FJS17a : ArXiv :1704.05353 [math.AP]
FJS17b : ArXiv :1707.06141 [gr-qc]

Introduction

- Use of symmetries to study Vlasov fields introduced by FJS15 for the massive/massless Vlasov fields.
- Purpose : Develop a perturbative approach of the stability of Minkowski spacetime, as a solution to the Einstein - massive Vlasov system, similar to the Lindblad-Rodniandki proof.
- FJS15 : Develop a commutator approach for the flat relativistic transport equation, and apply it to a conformally perturbation of Minkowski for massless particles (3+1) and massive particles (4+1).
- FJS17a : Improve the commutators approach $\rightarrow$ modified commutators to handle Vlasov-Nordström in (3+1).
- FJS17b : noinlinear stability of Minkowski space, solution to the Einstein-Vlasov system (massive particles) in 3+1.

Introduction

- First stability result : for $\Lambda = 1$, Friedrich ('85) ;
- First result for $\Lambda = 0$: work by Christodoulou-Klainerman (93) (vacuum), Lindblad-Rodnianski (05-10, vacuum + scalar field), Bieri-Zipser ('98, Maxwell), Loizelet (07, Maxwell), Speck ('09, Born-Infeld), Taylor ('16, massless Vlasov).
- Particularity of Lindblad-Rodnianski work : approach "purely perturbative" $\rightarrow$ pure vector field method.
- First result for a matter model for which the direct application of the vector field fails : LeFloch-Ma ('15) Klein-Gordon.
- For Vlasov, Taylor result extension of Dafermos work for the massless Vlasov in spherical symmetry : control of the characteristic system.
- FJS 17b and Lindblad-Taylor : both stability result for the Einstein-Vlasov, purely perturbative approach for the Einstein equations, both develop a vector-field method for the transport equation.

Introduction

- Here : wave gauge + hyperboloidal foliation (à la LeFLoch-Ma).
- Construct a commutator theory valid both for Einstein and simultaneously for the transport equation.
- Two difficulties : commutators do not live on the same space + need to take into account the perturbation $\rightarrow$ modified commutators.
- This talk focuses on the transport equation ; we build up on the (slightly rearranged) result by LeFloch-Ma.
- Purpose : report on the structure of the commutator formula for the transport equation to perform the estimates, and in particular the so-called null structure of the transport equation.

Organization of the talk

Formulation of the problem

- Consider on $\mathbb{R}^4_x$, g a metric of signature $(- + ++)$
- Define the perturbation, raising/lowering indices

$$\begin{aligned} h_{\alpha\beta} &= g_{\alpha\beta} - \eta_{\alpha\beta} \\ H^{\alpha\beta} &= g^{\alpha\beta} - \eta^{\alpha\beta} \\ h^{\alpha\beta} &= g^{\alpha\beta} - \eta^{\alpha\beta} + \mathcal{O}(h^2) \end{aligned}$$

- Vlasov matter : describes a statistical ensemble of particles of fixed mass 1 ; $f : \mathbb{R}^4_x \times \mathbb{R}^3_v \to \mathbb{R}$, satisfies a collisionless Boltzmann equation :

$$T_g f = g^{\alpha\gamma} v_\alpha \frac{\partial f}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial x^i} v_\alpha v_\beta \frac{\partial f}{\partial v_i} = 0$$

where v_0 is implicitly given by the formula

$$g^{\alpha\beta} v_\alpha v_\beta = -1$$

Formulation of the problem

- Einstein equations :

$$Ric(g)_{\alpha\beta} - \frac{1}{2}Scal(g)g_{\alpha\beta} = \underbrace{\int_v f v_\alpha v_\beta d\mu_v}_{\text{stress-energy tensor}}$$

- Fix the gauge : work in the wave coordinates : $\Box x^\alpha = 0$
- Einstein-Vlasov system reduces to a system on $\mathbb{R}^{3+1}_x \times \mathbb{R}^3_v$

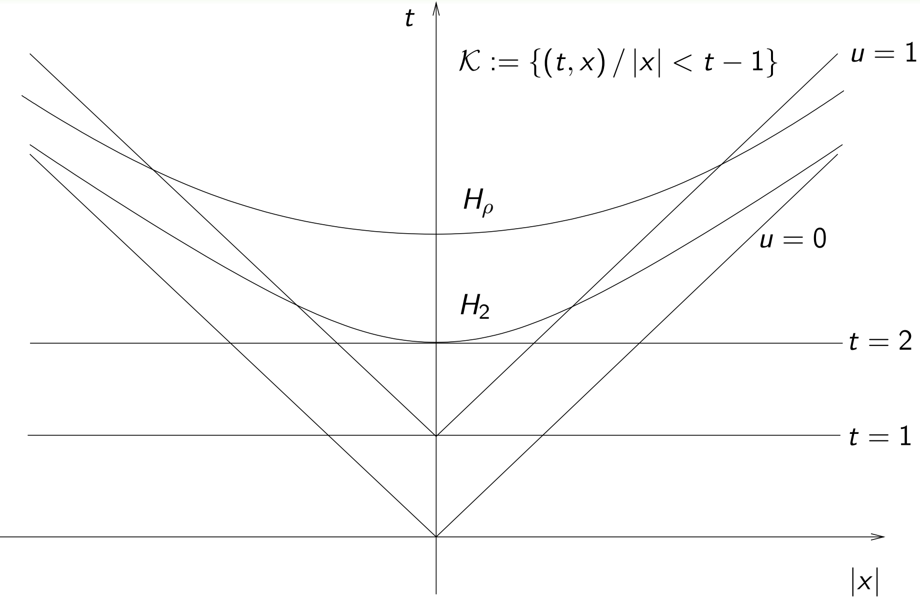
$$\begin{aligned} \tilde{\Box}_g h_{\alpha\beta} &= F_{\alpha\beta}(h, \partial h) - \int_v f(2v_\alpha v_\beta + g_{\alpha\beta}) d\mu_v \\ T_g(f) &= 0 \end{aligned}$$

- Problem : small data global existence + asymptotics ?

Hyperboloidal foliation

- Decay of massive fields : hyperloidal foliation used in the context of massive fields, by Hörmander and Klainerman (85 - 93) to deal with the different sets of commutators wave vs. Klein-Gordon.
- Lefloch-Ma : extensively used the hyperboloidal foliation to prove the the stability of Minkowski space as a solution to the Einstein-Klein-Gordon system.
- Here, we also use this hyperboloidal foliation to set up our Cauchy problem $\rightarrow$ Important for the weighted Sobolev estimates.
- On $\mathbb{R}^4$, with Cartesian coordinates, define

$$H_\rho = \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 | t^2 - |x|^2 = \rho^2\}, \text{ for } \rho \geq 1.$$


t
$\mathcal{K} := \{(t,x) \,/\, |x| < t-1\}$
$u = 1$
H_ρ
$u = 0$
H_2
$t = 2$
$t = 1$
$|x|$

Initial data

- Consider initial data, for which, at $t = 2$, compactly supported in space in the ball of Euclidean radius 1, for the metric and the distribution function $\leftarrow$ gluing to satisfy the constraint equations.
- No compactness assumption in velocity $\rightarrow$ the coupling near $\mathscr{I}$ is "on" $\rightarrow$ requires fine analysis in the outgoing direction.
- Furthermore, assume that, for ϵ small enough, $N \geq 14$, $q \geq 3$ and $\epsilon > 0$, that the ID satisfy

$$
\begin{gathered}
||h - \delta_E||_{H^N(|x|<1)} + ||k||_{H^{N-1}(|x|<1)} + m^2 \\
+||(1+|v|^2)^q f_0||_{W^{N+3,1}(T^\star\mathbb{R}^3)} + ||(1+|v|^2)^{q+2} f_0||_{W^{N+2,1}(T^\star\mathbb{R}^3)} \\
\leq \epsilon,
\end{gathered}
$$

The result

Theorem (FJS '17)

The Einstein Vlasov system, with data as on the previous slide admits a global solution in the future of the unit hyperboloid H_2 such that

$$\begin{aligned} \mathcal{E}_N[g](\rho) &\leq D_N \rho^{D_N\sqrt{\epsilon}} \\ E_{N-2,q+2}[f](\rho) &\leq D_N \rho^{D_N\sqrt{\epsilon}} \\ E_{N,q}[f][f](\rho) &\leq D_N \rho^{D_N\sqrt{\epsilon}} \end{aligned}$$

In particular, the Cauchy development of these initial data is future causally complete.

Remark : Klainerman-Sobolev estimates imply

$$|g-\eta| \lesssim (1+|t-r|)^{1/2}t^{-1}, |\partial g| \lesssim \rho^{-1}t^{-1/2}, \int_v |f|dv \lesssim t^{-3}, etc.$$

The Lindblad-Taylor result : Arxiv :1707.06079

- Foliation by the standard Minkowski time.
- Initial data : for the metric, non compact ; for the Vlasov field, compactly supported in space and velocity $\to$ no particle escape to null infinity.
- for Einstein : built up on the work by Lindblad, Lindblad-Rodnianski ;
- for Vlasov after Taylor, construct commutator, variation of the complete lifts, tailored to handle the full characteristic system of the Vlasov field, to the difference to our approach, where all perturbation terms are thrown in a junk that needs to be analyzed in detail.

Strategy of the proof

- Bootstrap argument for the energy for the metric.
- Find appropriate commutator $\hat{K}$, vector field on the mass shell : modified commutator wrt Minkowski space.
- Derive a general commutator formula with the vector fields $\hat{K}$.
- In this commutator formula,
  - Kill all the bad terms at the first order, by imposing the right equation in the modifiers.
  - Once the equation is chosen, established an n-th order commutator formula ;
  - in this higher order commutator formula, determine some null structures that need to be exploited to perform the energy estimates.
- Exploit the hierarchy of the proof of the first order commutator formula to devise a strategy to perform *braided* energy estimates.

The vector-field method for the wave equation

- Developed by Klainerman 85-90 for nonlinear wave equations.
- Recipe : a coercive energy estimate + commutator with the operator + weighted Sobolev estimates $\rightarrow$ pointwise estimates.
- Example for the wave equation $\Box\phi = 0$ on Minkowski space :

$$E[\phi](t) = \int_t (\partial_t \phi)^2 + |D\phi|^2 dx = constan$$

$$\text{Commutators } K : \partial_{x^\alpha}, x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha}, x^\alpha \partial_{x^\alpha}$$

$$\text{Klainerman Sobolev estimates} :$$

$$|(1+|t-r|)^{1/2}(1+|t+r|)\partial\phi| \leq C \sum_{|\alpha|\leq 2} E[K^\alpha \phi](t)$$

- Pointwise decay : small data global existence + decay for nonlinear wave.

Extension to the transport operator

- Extension in FJS15 to the flat transport (massive and massless) relativistic transport operator.
- Conserved quantity : conservation of matter.

$$\int_x \int_v \underbrace{W(v)}_{\text{weight in v}} |f| dxdv = \text{conserved}$$

- Klainerman-Sobolev inequalities for velocity averages : technical aspects once the right commutators are found.
- Essential nonetheless to work in the hyperboloidal foliation to get the sharpest decay for the velocity averages :

$$\int_v f dv \sim t^{-2}(1+|t-r|)^{-1} \text{in t-fol. vs } t^{-3} \text{ in the hyp. foliation}$$

- Difficulty : Simultaneous commutators with the wave equation and the transport equation, which is also commuting with the integration in the velocity variable.

Setup

- Setup : (M, g) Lorentzian manifold (oriented, time oriented). Consider $T^\star M$.
- Chart (U, x^α) on M ; natural extension to $T^\star U$: $(U, x^\alpha, v_\alpha = \partial_{x^\alpha})$.
- Geodesic spray / Liouville vector field (Hamiltonian vector field of $\frac{1}{2} g^{\alpha\beta} v_\alpha v_\beta$) :

$$T_g = g^{\alpha\gamma} v_\alpha \frac{\partial}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g^{\alpha\beta}}{\partial x^\gamma} v_\alpha v_\beta \frac{\partial}{\partial v_\gamma}$$

- Mass shell : $\mathcal{P} = \{(x, v) \in T^\star M | g^{\alpha\beta} v_\alpha v_\beta = -1\}$, with fiber $\mathcal{P}_x$.
- T_g is tangent to the mass shell.

A commutator formula

- Let K be a vector field, and X_K, the Hamiltonian vector field associated with $K^\alpha v_\alpha$;
- Trivial calculation :

$$[T_g, X_K] = -2\pi^{(Z)\alpha\gamma} v_\gamma e_\alpha + \nabla^\gamma \pi^{(Z)}_{\alpha\beta} v^\alpha v^\beta \frac{\partial}{\partial v^\gamma}$$

- Important facts, when K Killing :
  - X_K coincides with the complete lifts ;
  - X_K is tangent to the mass shell ;
  - X_K commutes with T_g ;
  - Morally, $f : \mathcal{P} \to \mathbb{R}$

$$\mathcal{L}_K \int_v f dv = \int_v X_K f dv.$$

Modified commutators

- On $(\mathbb{R}^4, \eta)$, consider, the vector fields :

$$Z = x_\alpha \partial_\beta - x_\beta \partial_{x^\alpha}, \; x^\alpha \partial_{x^\alpha}$$

- Lift these vectors onto $\mathcal{P}$:

$$\hat{Z} = x_0 \partial_i + x_i \partial_{x^0} + v_0 \partial_{v_i}, x^\alpha \partial_{x^\alpha}.$$

- Then modify the homogeneous vector fields :

$$Y = \hat{Z} + C^i X_i, \quad X_i \text{ well-chosen}$$

- FJS17a : X_i : bit of a Christofell symbols $\rightarrow$ decomposition in vertical/horizontal space.
- Here, because of good commutations relations + good decomposition + some null properties (see later) :

$$X_i = \partial_{x^i} + \frac{v_i}{\sqrt{1+|v|^2}} \partial_{x^0} = \frac{Z}{t} + \frac{v^i t - x_i \sqrt{1+|v|^2}}{\sqrt{1+|v|^2}} \partial_t$$

Deriving the commutator formula

- Treat the operator as :

$$T_g = w^{\alpha}\partial_{x^{\alpha}} + Q(g, \partial g, v)\partial_v f.$$

- Find a good decomposition of the operator so that we can perform estimates ; naively

$$[T_g, \hat{Z}] \text{ contains } \partial Z(g) \cdot \underbrace{\partial_v f}_{\sim t\partial_x f}$$

- Identify the good terms + absorb the bad ones in the commutators.
- Good terms : terms that have decay faster than ρ^{-1} (+loss).

The commutators

- the translation ∂_t
- the modified translation X_i
- the modified homogeneous vector fields

$$Y = \hat{Z} + C^i X_i$$

where $\hat{Z}$ are the modified complete lifts of $x_\alpha \partial_{x^\beta} - x_\beta \partial_{x^\alpha}$ and the pure scaling $x^\alpha \partial_{x^\alpha}$.

- Purpose : derive a formula for $[T_g, \widehat{K}]$, where $\widehat{K}$ is any of the vectors above, and, by imposing a transport equation on C, absorb the bad terms.

Mechanism for improvements

- Good derivatives of the metric components ;
- Wave gauge : better behaviour of h^{00} components : Klainerman-Sobolev estimates + sup-norm estimates provide :

$$\partial \underline{h}^{00} \sim \epsilon t^{-5/2} \rho^{\delta+1} \text{ instead of } \partial \underline{h}^{00} \sim \epsilon \rho^{-1} t^{-1/2} \rho^{\delta}.$$

- Presence of a t^{-1}-factor (in the decomposition of X_i, for instance).
- Presence of a hyperbolic weight $\mathfrak{z}_i = x_i w_0 + w_i t$: absorbed in the energy with a t/ρ-weight ;
- Existence of structure of the form :

$$\partial_{x^i}(k)\partial_{v^i}, \frac{w^\gamma}{w^0}\partial_{x^\gamma}.$$

Some mechanisms for improvements

Lemma

The following identities hold. For a regular function $k := k(t, x)$,

$$\partial_{x^i}(k)\partial_{v_i} = \underline{\partial}_i k \cdot \partial_{v_i} - \frac{x^i}{t}\partial_t k \cdot \frac{1}{w^0}\widehat{Z}_i + \partial_t k \frac{1}{w^0}\left(S + \frac{|x|^2 - t^2}{t}\partial_t\right).$$

The free transport operator can be rewritten as

$$\frac{w^\gamma}{w^0}\partial_{x^\gamma} = \frac{S}{t} + \frac{\underline{v}_i}{w^0}\partial_{x^i}.$$

Finally, ∂_{v_i} *can be rewritten as*

$$\begin{aligned} \partial_{v_i} &= \frac{\hat{Z}_i}{w^0} - \frac{t}{w^0}X_i + t\frac{\underline{v}_i}{(w^0)^2}\partial_t \\ &= \frac{\hat{Z}_i}{w^0} - \frac{t}{w^0}X_i + \frac{\mathfrak{z}_i}{(w^0)^2}\partial_t. \end{aligned}$$

Energy for the matter distribution

- Obtained by contracting the normal with the hyperboloid with the stress energy tensor.
- Use the identities :

$$\begin{aligned}\nu_\rho^\alpha w_\alpha w_0 &= \frac{t}{2\rho}\left(\frac{\rho^2}{t^2}(w^0)^2 + 1 + \sum_{i=1}^{3}(\underline{v}_i)^2\right)\\ &= \frac{t}{2\rho}\left(\frac{\rho^2}{t^2}(w^0)^2 + 1 + \sum_{i=1}^{3}\mathfrak{z}_i^2 t^{-2}\right)\\ &= \frac{t}{2\rho}\left(\left(w_0 + \frac{x^i}{t}w_i\right)^2 + \sum_{1\le i<j\le 3}\left(\frac{x^i w_j - x^j w_i}{t}\right)^2 + \frac{\rho^2}{t^2}w_0^2\right.\end{aligned}$$

where $\mathfrak{z}_i$ is the hyperbolic weight $\mathfrak{z}_i = w_0 x_i + w_i t$ and $\underline{v}_i = \frac{\mathfrak{z}_i}{t}$.

Energy for the matter distribution

- The flat energy at order 0 :

$$E[f] = \int_{H_\rho} \int_v |f| \frac{1}{2w^0} \left(\frac{\rho^2}{t^2} (w^0)^2 + 1 + \sum_{i=1}^{3} \underline{v}_i^2 \right) dvdx. \quad (1)$$

- Add multiplication by the weight w^0, and the commutators to obtain the energy at higher order :

$$E_{N,q}[f](\rho) := \sum_{|\alpha| \leq N} E\left[(1 + |v|^2)^{q/2}] \widehat{K}^\alpha f \right] (\rho), \quad (2)$$

where $\widehat{K}$ denotes any vector fields among ∂_t, X_i or the Y vector fields.

Braided energy estimates at first order

- Direct use of energy estimates fails by lack of decay.
- Following the proof of the commutator formula, done in several steps, braided energy estimates :

$$
\begin{aligned}
E[\partial_t f](\rho) - E[\partial_t f](2) &\leq C \int_2^\rho s^{-3/2+\delta} E[\hat{K} f](s) + s^{\delta/2-1} E[Xf](s \\
E[Xf](\rho) - E[Xf](2) &\leq C\sqrt{\epsilon} \int_2^\rho s^{-3/2+\delta} E[\hat{K} f](s) ds \\
E[Yf](\rho) - E[Yf](2) &\leq C\sqrt{\epsilon} \int_2^\rho s^{-3/2+\delta} E[\hat{K} f](s) ds \\
&\quad + \sqrt{\epsilon} \int_2^\rho s^{\delta-1} \left(E[Xf](s) + E[\partial_t f](s) \right) ds
\end{aligned}
$$

A word about the coupling in the Einstein equations

- For the Einstein part, follow LeFloch-Ma.
- For the transport part : require to prove decay of L^2 norms of for the matter distribution $\rightarrow$ tools where developed in FJS15. Relies on a representation of solutions of the equation satisfied by f^2.
- At the top order for g, $\rightarrow$ top order for f : obstacles to performing the estimates ; problem solved by an integration in the velocity variable to decrease the order of the commutator.

Concluding remarks

- The approach here is not depending on the compactly support data in space.
- The same analysis can be extended to massless particles $\rightarrow$ Analysis is expected to be simpler.
- Of course, should be extended to (classical) Vlasov-Maxwell, and other equations.